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Schrödinger-type Equation with the Data in Morrey Spaces

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ABSTRACT

In this paper we investigate the regularity properties of Schrödinger-type equations with the data in certain Morrey spaces. Under some assumptions, we obtain that if the data is not in certain Morrey space, then the solution is also not in certain Morrey spaces. Hence, if the data which acts on the system does not balance local regularity and global decay, then the solution does not either.

Keywords:
Schrödinger-type equation; Data; Morrey spaces

1. Introduction

Quantum mechanics stands as one of the most remarkable intellectual achievements of recent decades, significantly expanding the field of physics. This theory forms a foundational framework for explaining phenomena at the microscopic level, including the behavior of subatomic particles and the internal structure of matter. Within this context, partial differential equations—such as the Schrödinger and Klein-Gordon equations—play a vital role in analyzing the fundamental dynamics of quantum systems. A central focus in such studies is the investigation of the regularity properties of their solutions, which offer valuable insights into the stability and evolution of quantum states. The origins of quantum theory can be traced back to Max Planck's groundbreaking work on blackbody radiation in the early 20th century [1], for which he was later awarded the Nobel Prize.

We shall consider the Schrödinger-type Equations

$$\left[V^{\gamma}(-\Delta+V)^{-\beta}\right](u)=f \tag{1}$$

where $0 \le \gamma \le \beta \le 1$ and

$$\left[V^{\gamma}\nabla(-\Delta+V)^{-\beta}\right](u)=f \tag{2}$$

where $0 \le \gamma \le \frac{1}{2} \le 1, \beta-\gamma \ge \frac{1}{2}$, and V is nonnegative potential belonging to the reverse Holder class B_∞. The function f is called data which describes external forces acting on the system. In this paper, we prefer to use the notion of forcing function.

The Schrödinger equation governs the behavior of particles in quantum spaces, capturing complex dynamics that are central to quantum theory. Several recent studies such as those by Geng et al. [2], Hossein et al. [3], Ibrahim et al. [4], Ibrahim & Baleanu [5], Litu et al. [6], and Rafiq et al. [7] have contributed to a deeper understanding of this equation. Due to its intricate structure, finding explicit solutions for the function u u is often challenging. Nevertheless, mathematical analysis offers powerful tools for examining various qualitative properties of the solutions.

In this study, we focus on a specific question: What happens to the behavior of the solution u when the function f in equations (1) and (2) lacks a balance between local regularity and global decay? In other words, we aim to investigate how the solution behaves when the function f exhibits irregularities both at small (local) scales and at infinity (global scales), making it difficult to control or predict its influence on u.

The concepts of local and global behavior can be effectively analyzed using the framework of Morrey spaces. These spaces were initially introduced by Charles Bradfield Morrey in 1938 as a tool for investigating the local properties of solutions to certain elliptic partial differential equations [8]. Morrey spaces are regarded as a generalization of Lebesgue spaces, offering a broader setting for capturing both local regularity and global integrability characteristics.

Morrey spaces have been extensively utilized to examine the behavior of solutions to various partial differential equations through the application of operators studied in mathematical analysis, particularly in harmonic analysis. Commonly employed operators include the Hardy–Littlewood maximal operator, Riesz potential, Calderón–Zygmund operators, and the Fourier transform. For a broader treatment of operator boundedness on these spaces, readers are referred to the works of Ramadana & Gunawan [9-10] and Samko [11].

In this study, we examine the solutions of equations where f belongs to a Morrey space, focusing particularly on the regularity properties of the solutions. Our investigation is conducted within a broader framework, namely the generalized Morrey spaces. We establish the regularity of the solutions to Schrödinger equations (1) and (2) by employing analytical tools from mathematical analysis, specifically the fractional maximal operator M_α. We begin by proving the boundedness of this operator on generalized weighted Morrey spaces, and then apply the results to the equations under consideration.

2. Preliminaries: Some Definitions and Notations

We write $\mathbb{R}^+ = (0, \infty)$. We shall use the Euclidean spaces $\mathbb{R}^n$ endowed by the usual metric and measure defined on it. For $z \in \mathbb{R}^n$ and $t > 0$, we denote by $B(z,t)$ the ball centered at z with radius of $t > 0$. Moreover, $|E|$ denotes the Lebesgue measure of the set $E \subseteq \mathbb{R}^n$.

For $1 < p < \infty$ and $\varphi: \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}^+$. Morrey spaces L^p_φ is a set of any locally integrable function $f \in L^p_q$ such that the norm

$$\|f\|_{L^p_q} = \sup_{z\in\mathbb{R}^n, t>0} \frac{1}{\varphi(z,r)} \frac{1}{t^{\frac{n}{p}}} \left(\int_{B(z,t)} |f|^p\right)^{\frac{1}{p}} \simeq \sup_{z\in\mathbb{R}^n, t>0} \frac{1}{\varphi(z,r)} \frac{1}{|B(z,r)|^{\frac{1}{p}}} \left(\int_{B(z,t)} |f|^p\right)^{\frac{1}{p}}$$

is finite. We say $V \in B_\infty$ if there is a positive constant $C > 0$ for which

$$\|V\|_{L^\infty(B)} \leq \mathrm{C} \frac{1}{|B|} \int_B V(z)\, dz$$

for any ball B in $\mathbb{R}^n$ [12]. We consider the two Schrödinger operators L_1 and L_2 where

$$L_1(u) = \left[V^\gamma(-\Delta + V)^{-\beta}\right](u)$$

and

$$L_2(u) = \left[V^{\gamma}\nabla(-\Delta + V)^{-\beta}\right](u).$$

For those interested in exploring further developments and applications of Schrödinger operators, several recent studies provide valuable insights. Notably, the works of Akbulut et al. [13], Ambrosio [14], Dasgupta et al. [15], Dewan [16], and Fabris et al. [17] offer comprehensive discussions and analyses on various aspects of Schrödinger operators. These include theoretical advancements, new solution techniques, and applications in mathematical physics. Readers are encouraged to consult these references to gain a deeper understanding of the current research landscape and ongoing progress in this area of mathematical analysis.

3. Results and Discussion

We denote S_0^{∞} by the set of any infinitely differentiable function on $\mathbb{R}^n$ with compact support. For any ball B, we denote $l(B)$ by the radius of B and $c(B)$ denotes the center of B. Moreover, L_{loc}^p denotes the set of any functions f such that $f \cdot \mathcal{X}_B \in L^p$ for all balls B on $\mathbb{R}^n$ where $\mathcal{X}_B$ is the characteristic function of the ball B.

We have the following useful theorems. The theorems provide the relation between the Schrödinger operator (L_1 and L_2) and M_α which is inspired us to use the method via boundedness operator M_α to answer the question presented in the section 1.

Theorem 1 [18]. *Suppose $V \in B_\infty$ and $0 \le \gamma \le \beta \le 1$. Then,*

$$|L_1 u| \lesssim M_\alpha u, \qquad u \in S_0^{\infty}$$

where $\alpha = 2(\beta - \gamma)$.

Theorem 2 [18]. *Suppose $V \in B_\infty$ and $0 \le \gamma \le \frac{1}{2} \le 1, \beta - \gamma \ge \frac{1}{2}$. Then,*

$$|L_2 u| \lesssim M_\alpha u, \qquad u \in S_0^{\infty}$$

where $\alpha = 2(\beta - \gamma) - 1$.

The following theorem was proved by Ramadana & Gunawan [9].

Theorem 3. *Let $1 \le p < q < \infty$. Then,*

$$\|M_\alpha(u)\|_{L^q(B)} \lesssim l(B)^{\frac{n}{q}} \int_{l(B)}^{\infty} t^{\frac{n}{p}-\frac{n}{q}} \|u\|_{L^p(B(c(B),t))} \frac{dt}{t}$$

for any ball B and $u \in L_{loc}^p$.

Theorem 4. *Let $0 < \alpha < n$, $1 \le p < \infty$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. Suppose the functions φ_1 and φ_2 satisfy*

$$\int_r^{\infty} \varphi_1(z,t) \frac{dt}{t^{1-\frac{\alpha}{n}}} \lesssim \varphi_2(z,r), \qquad (z,r) \in \mathbb{R}^n \times \mathbb{R}^+.$$

Then, M_α is bounded from $L_{\varphi_1}^p$ to $L_{\varphi_2}^q$.

In addition to the previous theorem, we establish an alternative condition that ensures the boundedness of the fractional maximal operator M_α on generalized Morrey spaces. This result is formulated in the following theorem.

Theorem 5. *Let $0 < \alpha < n$, $1 \le p < \infty$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. Suppose the functions φ_1 and φ_2 satisfy*

$$\sup_{r<t<\infty} \varphi_1(z,t) t^{\frac{n}{p}-\frac{n}{q}} \lesssim \varphi_2(z,r), \qquad (z,r) \in \mathbb{R}^n \times \mathbb{R}^+.$$

Then, M_α is bounded from $L_{\varphi_1}^p$ to $L_{\varphi_2}^q$.

Proof. We first find local estimate for the fractional maximal function M_α which is similar to the local estimate stated in Theorem 3. To do so, we let $u \in L_{\varphi_1}^p$, $z \in \mathbb{R}^n$, and $r > 0$. We split the function u to the form $f = u_1 + u_2$ where $u_1 = u \cdot \mathcal{X}_{B(z,2r)}$. It is well-know that M_α is bounded from L^p to L^q, then

$$\|M_\alpha(u_1)\|_{L^q(B(a,r))} \leq \|M_\alpha(u_1)\|_{L^q} \lesssim \|u_1\|_{L^p} = \|u\|_{L^p(B(z,2r))}$$

Hence,

$$\|M_\alpha(u_1)\|_{L^q(B(z,r))} \lesssim \|u\|_{L^p(B(z,2r))} \lesssim t^{\frac{n}{p}} \sup_{r<t<\infty} t^{-\frac{\alpha}{n}} \|u\|_{L^p(B(z,t))}.$$

Next, we find the estimate for the norm of u_2 over the ball $B(z,r)$ under L^q. We thus let $x \in B(z,r)$ and $t > 0$. If $y \in B(z,r)$ and $y \in B(z,2r)^c$, then we may obtain $r = 2r - r \leq |y - z| - |z - x| \leq |y - x| < t$ which implies the integral of $|u|$ over $B(x,t) \cap B(a,2r)^c$ equls 0 for $t \leq r$. Moreover, we also obtain that $B(x,t) \cap B(z,2r)^c \subseteq B(z,2t)$ for $r < t$. In fact, $|y - a| \leq |y - x| + |x - a| \leq t + r < 2t$. Hence, the following estimate hold by Holder inequality.

$$\begin{aligned}
M_\alpha(u_2)(x) &= \max\left(\sup_{t>r} \frac{1}{|B(x,t)|^{1-\frac{\alpha}{n}}} \int_{B(x,t)} |u_2|,\ \sup_{0<t\leq r} \frac{1}{|B(x,t)|^{1-\frac{\alpha}{n}}} \int_{B(x,t)} |u_2|\right) \\
&= \sup_{t>r} \frac{1}{|B(x,t)|^{1-\frac{\alpha}{n}}} \int_{B(x,t)} |u_2| = \sup_{t>r} \frac{1}{|B(x,t)|^{1-\frac{\alpha}{n}}} \int_{B(x,t)\cap B(a,2r)^c} |u| \\
&\leq \sup_{t>r} \frac{1}{|B(x,t)|^{1-\frac{\alpha}{n}}} \int_{B(z,2t)} |u| \cong \sup_{t>2r} \frac{1}{|B(z,t)|^{1-\frac{\alpha}{n}}} \int_{B(z,t)} |u| \\
&\leq \sup_{t>2r} \frac{1}{|B(z,t)|^{1-\frac{\alpha}{n}}} \|u\|_{L^p(B(z,t))} \|1\|_{L^{p'}(B(z,t))} \lesssim \sup_{t>r} \frac{1}{|B(z,t)|^{\frac{1}{q}}} \|u\|_{L^{p,}(B(z,t))}.
\end{aligned}$$

From the last inequality, we take the norm of L^q on $M_\alpha(u_2)$ over the ball $B(z,r)$ to obtain

$$\|M_\alpha(u_2)\|_{L^q B(z,r)} \lesssim r^{\frac{n}{q}} \sup_{t>2r} \frac{1}{t^{\frac{n}{q}}} \|u\|_{L^p(B(z,t))}.$$

We combine the obtained estimate for $M_\alpha(u_1)$ and $M_\alpha(u_2)$ and then use the linearity properties of M_α on Lebesgue spaces,

$$\|M_\alpha(u)\|_{L^q B(z,r)} \lesssim r^{\frac{n}{q}} \sup_{t>2r} \frac{1}{t^{\frac{n}{q}}} \|u\|_{L^p(B(z,t))}.$$

Hence, we use the definition of the norm of generalized Morrey spaces,

$$\begin{aligned}
\|M_\alpha(u)\|_{L^p_{\varphi_2}} &= \sup_{z\in\mathbb{R}^n, r>0} \frac{1}{\varphi_2(z,r)} \frac{1}{t^{\frac{n}{q}}} \|M_\alpha(u)\|_{L^q B(z,r)} \\
&\lesssim \sup_{z\in\mathbb{R}^n, r>0} \frac{1}{\varphi_2(z,r)} \frac{1}{r^{\frac{n}{q}}} r^{\frac{n}{q}} \sup_{t>r} \frac{1}{t^{\frac{n}{q}}} \|u\|_{L^p(B(z,t))} \\
&= \sup_{z\in\mathbb{R}^n, r>0} \frac{1}{\varphi_2(z,r)} \sup_{t>r} \frac{1}{t^{\frac{n}{q}}} \|u\|_{L^p(B(z,t))} \\
&= \sup_{z\in\mathbb{R}^n, r>0} \frac{1}{\varphi_2(z,r)} \sup_{t>r} \frac{\varphi_1(z,t)}{\varphi_1(z,t)} \frac{1}{t^{\frac{n}{q}}} \|u\|_{L^p(B(z,t))} \\
&\leq \|u\|_{L^p_{\varphi_1}} \sup_{z\in\mathbb{R}^n, r>0} \frac{1}{\varphi_2(z,r)} \sup_{t>r} \varphi_1(z,t) \frac{t^{\frac{n}{p}}}{t^{\frac{n}{q}}} \\
&\lesssim \|u\|_{L^p_{\varphi_1}}.
\end{aligned}$$

This proves that M_α is bounded from $\mathcal{M}^{p,w^p}_{\varphi_1}$ to $\mathcal{M}^{q,w^q}_{\varphi_2}$ ∎

Using these theorems, we have the following theorem which present the boundedness of Schrödinger operator on the Morrey spaces.

Theorem 6. *Suppose* $V \in B_\infty$ *and* $0 \le \gamma \le \beta \le 1$. *Let* $\alpha = 2(\beta - \gamma) - 1$, $1 \le p < \infty$, *and* $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. *Suppose the functions* φ_1 *and* φ_2 *satisfy one of the following conditions:*

1. *For* $(z,r) \in \mathbb{R}^n \times \mathbb{R}^+$,
$$\int_r^\infty \varphi_1(z,t) \frac{dt}{t^{1-\frac{\alpha}{n}}} \lesssim \varphi_2(z,r).$$
2. *For* $(z,r) \in \mathbb{R}^n \times \mathbb{R}^+$,
$$\sup_{r<t<\infty} \varphi_1(z,t) t^{\frac{n}{p}-\frac{n}{q}} \lesssim \varphi_2(z,r),$$

Then, L_1 *is bounded from* $L^p_{\varphi_1}$ *to* $L^q_{\varphi_2}$.

Proof. We note by Theorem 1, Theorem 4, and Theorem 5 that
$$\|L_1(u)\|_{L^q_{\varphi_2}} \lesssim \|M_\alpha(u)\|_{L^q_{\varphi_2}} \lesssim \|u\|_{L^p_{\varphi_1}}$$
which prove the boundedness of L_1 from $L^p_{\varphi_1}$ to $L^q_{\varphi_2}$ ■

Theorem 7. *Suppose* $V \in B_\infty$ *and* $0 \le \gamma \le \frac{1}{2} \le 1, \beta - \gamma \ge \frac{1}{2}$. *Let* $\alpha = 2(\beta - \gamma) - 1$, $1 \le p < \infty$, *and* $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. *Suppose the functions* φ_1 *and* φ_2 *satisfy one of the following conditions:*

1. *For* $(z,r) \in \mathbb{R}^n \times \mathbb{R}^+$,
$$\int_r^\infty \varphi_1(z,t) \frac{dt}{t^{1-\frac{\alpha}{n}}} \lesssim \varphi_2(z,r).$$
2. *For* $(z,r) \in \mathbb{R}^n \times \mathbb{R}^+$,
$$\sup_{r<t<\infty} \varphi_1(z,t) t^{\frac{n}{p}-\frac{n}{q}} \lesssim \varphi_2(z,r),$$

Then, L_2 *is bounded from* $L^p_{\varphi_1}$ *to* $L^q_{\varphi_2}$.

Proof. We note by Theorem 2, Theorem 4, and Theorem 5 that
$$\|L_1(u)\|_{L^q_{\varphi_2}} \lesssim \|M_\alpha(u)\|_{L^q_{\varphi_2}} \lesssim \|u\|_{L^p_{\varphi_1}}$$
which prove the boundedness of L_1 from $L^p_{\varphi_1}$ to $L^q_{\varphi_2}$ ■

Suppose the Schrödinger equation (1) and (2) again.The following two theorems are our main results regarding regularity result for the solution of the equation.

Theorem 8. *Suppose* $V \in B_\infty$ *and* $0 \le \gamma \le \beta \le 1$. *Let* $\alpha = 2(\beta - \gamma) - 1$, $1 \le p < \infty$, *and* $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. *Suppose the functions* φ_1 *and* φ_2 *satisfy one of the following conditions:*

1. *For* $(z,r) \in \mathbb{R}^n \times \mathbb{R}^+$,
$$\int_r^\infty \varphi_1(z,t) \frac{dt}{t^{1-\frac{\alpha}{n}}} \lesssim \varphi_2(z,r).$$
2. *For* $(z,r) \in \mathbb{R}^n \times \mathbb{R}^+$,
$$\sup_{r<t<\infty} \varphi_1(z,t) t^{\frac{n}{p}-\frac{n}{q}} \lesssim \varphi_2(z,r),$$

If u *is the solution of (1) and* $u \in L^{p,}_{\varphi_1}$, *then* $\|f\|_{L^q_{\varphi_2}} \lesssim \|u\|_{L^p_{\varphi_1}}$.

Proof. Suppose the equation (1) and $u \in L^p_{\varphi_1}$ is the solution of the equation. By Theorem 6, then
$$\|f\|_{L^q_{\varphi_2}} = \|L_1(u)\|_{L^q_{\varphi_2}} \lesssim \|u\|_{L^p_{\varphi_1}}$$
that proves the theorem ■

Theorem 9. *Suppose* $V \in B_\infty$ *and* $0 \le \gamma \le \frac{1}{2} \le 1, \beta - \gamma \ge \frac{1}{2}$. *Let* $\alpha = 2(\beta - \gamma) - 1$, $1 \le p < \infty$, *and* $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. *Suppose the functions* φ_1 *and* φ_2 *satisfy one of the following conditions:*

1. *For* $(z, r) \in \mathbb{R}^n \times \mathbb{R}^+$,
$$\int_r^\infty \varphi_1(z,t) \frac{dt}{t^{1-\frac{\alpha}{n}}} \lesssim \varphi_2(z,r).$$
2. *For* $(z, r) \in \mathbb{R}^n \times \mathbb{R}^+$,
$$\sup_{r<t<\infty} \varphi_1(z,t) t^{\frac{n}{p}-\frac{n}{q}} \lesssim \varphi_2(z,r),$$

If u *is the solution of (2) and* $u \in L^{p}_{\varphi_1}$, *then* $\|f\|_{L^q_{\varphi_2}} \lesssim \|u\|_{L^p_{\varphi_1}}$.

Proof. Suppose the equation (2) and $u \in L^p_{\varphi_1}$ is the solution of the equation. By Theorem 7, then
$$\|f\|_{L^q_{\varphi_2}} = \|L_1(u)\|_{L^q_{\varphi_2}} \lesssim \|u\|_{L^p_{\varphi_1}}$$
that proves the theorem ∎

Generally, we obtain that $\|f\|_{L^q_{\varphi_2}} \lesssim \|u\|_{L^p_{\varphi_1}}$ with different value for the parameter in the equations. It means that if the solution u in $L^p_{\varphi_1}$, the data f is forced to be in the Morrey spaces $L^q_{\varphi_2}$. In the other words, if the data is not in certain Morrey space, then the solution u is not in certain Morrey spaces. Hence, if the data which acts on the system does not balance local regularity and global decay, then the solution u does not either. This answers the research question.

4. Conclusion

The solution of Schrödinger-type equation have relation with the data in the equation, namely if the data is not in a Morrey spaces, then the solution of the equation is also not in certain Morrey spaces.

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